1. Give an example of 3 events A, B, C which are pairwise independent but not independent. Hint: find an example where whether C occurs is completely determined if we know whether A occurred and whether B occurred, but completely undetermined if we know only one of these things.
2. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?

Solution

Let’s solve each problem step by step:

### Problem 1: Example of Pairwise Independent Events That Are Not Independent

\*\*Solution:\*\*

We need to find three events \( A \), \( B \), and \( C \) such that:

1. \( A \) and \( B \) are independent.

2. \( A \) and \( C \) are independent.

3. \( B \) and \( C \) are independent.

4. However, \( A \), \( B \), and \( C \) are not independent together.

\*\*Example:\*\*

Consider flipping two fair coins:

- Let \( A \) be the event that the first coin is heads.

- Let \( B \) be the event that the second coin is heads.

- Let \( C \) be the event that the number of heads is even.

Now, let’s check the conditions:

- \( P(A) = P(B) = P(H) = \frac{1}{2} \) (since each coin flip is independent and the probability of heads is \( \frac{1}{2} \)).

- \( P© = P(\text{even number of heads}) = \frac{1}{2} \).

\*\*Pairwise Independence:\*\*

- \( P(A \cap B) = P(\text{HH}) = \frac{1}{4} \), and \( P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \). So, \( A \) and \( B \) are independent.

- \( P(A \cap C) = P(\text{H and even number of heads}) = P(\text{HT}) = \frac{1}{4} \), and \( P(A) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \). So, \( A \) and \( C \) are independent.

- \( P(B \cap C) = P(\text{T and even number of heads}) = P(\text{HT}) = \frac{1}{4} \), and \( P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \). So, \( B \) and \( C \) are independent.

\*\*Not Independent Together:\*\*

- \( P(A \cap B \cap C) = P(\text{HH}) = \frac{1}{4} \).

- \( P(A) \times P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \).

Since \( \frac{1}{4} \neq \frac{1}{8} \), the events \( A \), \( B \), and \( C \) are not independent together.

### Problem 2: Probability of Remaining Marble Being Green

\*\*Solution:\*\*

We have the following scenario:

- A bag contains one marble which is either green or blue, with equal probabilities (\( \frac{1}{2} \) each).

- A green marble is added to the bag.

- A random marble is taken out, and it is green.

We need to find the probability that the remaining marble is green.

Let:

- \( G\_1 \): The initial marble is green.

- \( B\_1 \): The initial marble is blue.

- \( G\_2 \): The remaining marble is green.

- \( G\_3 \): The marble taken out is green.

We are asked to find \( P(G\_2 | G\_3) \).

By Bayes’ theorem:

\[

P(G\_2 | G\_3) = \frac{P(G\_3 | G\_2) \cdot P(G\_2)}{P(G\_3)}

\]

First, calculate \( P(G\_2) \):

* \( P(G\_2) = P(G\_1) \times P(G\_3 | G\_1) + P(B\_1) \times P(G\_3 | B\_1) \).

Where:

- \( P(G\_1) = \frac{1}{2} \).

- \( P(B\_1) = \frac{1}{2} \).

- \( P(G\_3 | G\_1) = \frac{1}{2} \) (if the first marble was green, the second marble was also green).

- \( P(G\_3 | B\_1) = 1 \) (if the first marble was blue, the remaining one is green).

Thus:

\[

P(G\_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}

\]

Next, calculate \( P(G\_3) \):

- \( P(G\_3) = P(G\_1) \times P(G\_3 | G\_1) + P(B\_1) \times P(G\_3 | B\_1) \).

- \( P(G\_3) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \).

Finally:

\[

P(G\_2 | G\_3) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}

\]

So, the probability that the remaining marble is green is \( \frac{1}{3} \).